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Development of Prediction Model Using Ensemble Forecast Assimilation in Nonlinear Dynamical System

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Abstract

In this study a new type of ensemble forecast assimilation technique is developed in order to improve the forecast skill in the nonlinear dynamical system. The forecast assimilation is an analysis technique in which a true value contained in each ensemble forecast is accumulated into a single assimilated forecast such as a data assimilation. For the experiments, we used a Lorenz model, and a Kalman filter is applied for the forecast assimilation.

The experiments are started by calculating 101 members of the ensemble forecast in which the initial error with Gaussian distribution is superimposed around the truth, and one of the members is arbitrarily selected as a control forecast. The experiments of the forecast assimilation are repeated 5000 times for different sectors of the solution trajectory to obtain the statistical significance of the results. The distribution of the ensemble members is stretched by a linear error growth at the beginning of the forecast. After that, the nonlinear effect becomes dominant to distort the distribution. The forecast assimilation is then started when the errors of the ensemble forecasts have grown to a certain threshold. It is demonstrated that the forecast skill of the assimilated forecast is always superior to the control forecast. In the range of the small root mean square error (RMSE) of the ensemble forecast, the skill of the assimilated forecast is inferior to the ordinary ensemble mean. However, for the sufficiently large RMSE before the saturation, it is shown that the skill of the assimilated forecast is superior to the ensemble mean. The result suggests that the forecast assimilation is one of the viable approaches to the medium or extended range forecast.

1. Introduction

A discovery of chaos in an atmospherical dynamics by Lorenz (1963) indicates that a deterministic forecast is limited by a small error in the initial condition, even if we can have a perfect prediction model. It is caused by the fact that the atmosphere has the property of the inherent nature of the nonlinearity and instability. So, the atmospheric motion is essentially deterministic in the early stage, statistical in a moment after the time integration, and unpredictable at last. The limit of predictability for such an atmosphere is estimated as about two weeks (Lorenz 1982; Toth 1991), and the

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doubling time of the small error is estimated as about 1.9 days by Nohara and Tanaka (2001). Errico et al. (2002) commented that knowing the limit of the atmospheric predictability is important for setting a reasonable goal of the numerical weather prediction.

In order to extend the predictable period, an ensemble of numerical forecasts from slightly perturbed initial conditions is used for the medium range forecasts at many operational weather forecasting centers. Since the ensemble forecast is based on a probabilistic weather prediction, it is necessary to create a probability density function as diversely as possible for the ensemble members. Therefore, at the European Center for Medium-Range Weather Forecasts (ECMWF), the ensemble forecast system is constructed by an initial perturbation that is a linear combination of singular vectors (Molteni et al. 1996; Buizza et al. 2000). The singular vectors specify the directions of the greatest growth of the linearized system over a predetermined time interval. At the National Centers for Environmental Prediction (NCEP), a breeding method is introduced for the ensemble forecast system in which the initial perturbation is a linear combination of bred vectors (Toth and Kalney 1993, 1997). The bred vectors are perturbations created in directions where past forecast errors have grown rapidly. These ensemble forecasts have contributed to the progress of the medium range forecast in the 1990's.

Recently, a multi-model ensemble system has been suggested by Palmer et al. (2000). The ensemble members are constructed by many operational forecasts from different operational centers that run competitive state-of-the-art operational analysis and model forecasts. The ensemble average of the multi-model ensemble is more skillful than the best individual forecasts (Fritsch et al. 2000). Krishnamurti et al. (1999) has shown that if the multi-model ensemble includes correction of the systematic errors by regression (called superensemble), the forecast skill is significantly improved.

Using these ensemble forecasts, some statistical analyses like the ensemble mean perform well in the nonlinear system. Certainly, if the ensemble members distribute hyper cubic or ellipsoid around the truth, the location of the ensemble mean indicates the truth, and it becomes the best forecast. Nevertheless, the actual distribution is folded by the nonlinear effect as the forecast progresses. Then, the ensemble mean is detached from the center of the distribution of the ensemble members, since the ensemble mean is only the average of the ensemble members. If one statistical analysis includes not only the temporal result of the ensemble members, but also the time evolution of the forecast error, it may be expected that the skill of the new analyzed forecast is better than the ordinary ensemble mean. For the statistical analysis, the ensemble members are accumulated into a single forecast by a Kalman filter (Kalman and Bucy 1961), as in the atmospheric and oceanic data assimilation. This is called forecast assimilation. The Kalman filter assimilates the explicit description of the evolution of the forecast error, so it is competent for the forecast assimilation.

The purpose of this study is to construct a new technique of the forecast assimilation using the ensemble members in order to improve the forecast skill. In order to examine the performance of the forecast assimilation technique, we use a simple dynamical system of the Lorenz model (Lorenz 1963), which has been comprehensively studied in terms of chaotic behavior, and the nonlinear system (Sparrow 1982; Mukougawa et al. 1991). For the assimilation of the ensemble members, we use the Kalman filter formulated by Kalman and Bucy (1961).

The experimental design is described in Section 2, where the forecast assimilation is presented in detail. In Section 3, the result of the assimilated forecast is presented to compare the forecast skill with the ensemble mean forecast. The summary and discussion are given in Section 4.

2. Experimental design

In this section, we describe the experimental design of the forecast assimilation in detail. The forecast assimilation is a new technique for improving the weather forecast skill. In this study, the Lorenz model (Lorenz 1963) is used to examine the effect of the forecast assimilation.

2.1 Forecast assimilation

The forecast error of the ensemble mean is generally smaller than the single forecast. Con-

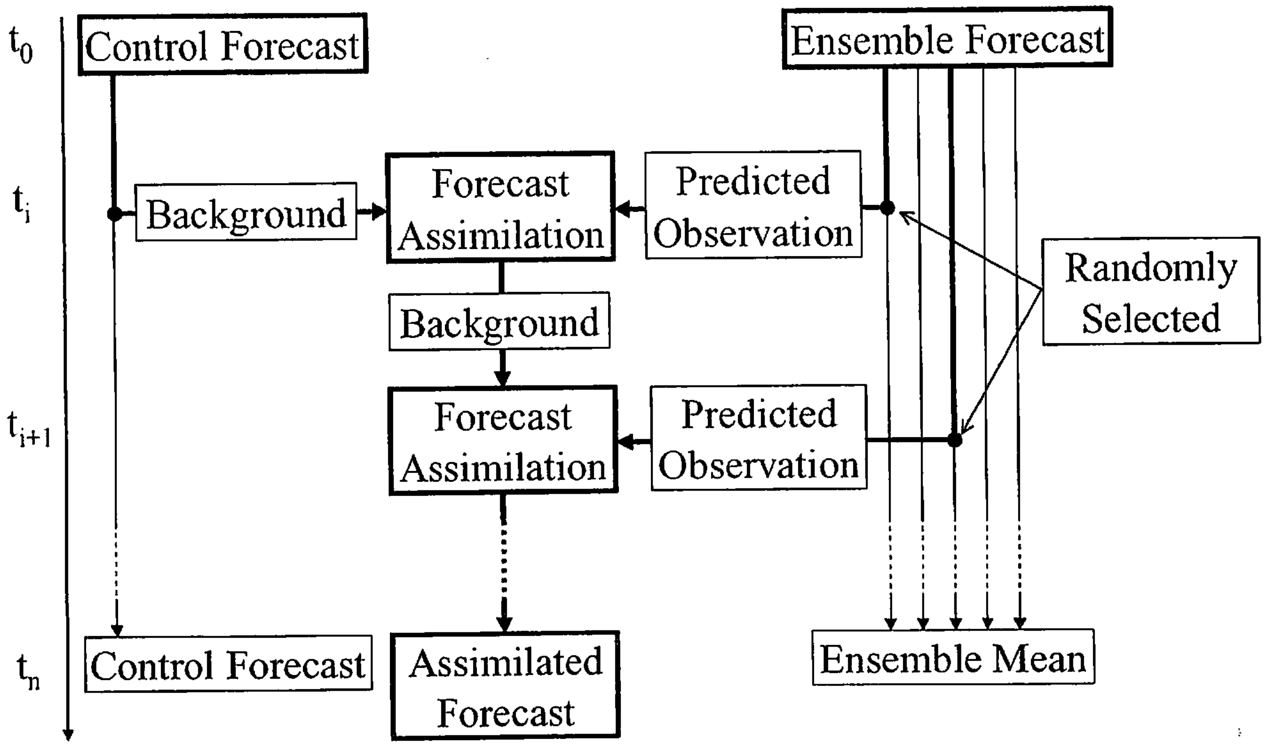

Fig. 1. Schematic flowchart of the forecast assimilation. The bold arrows and boxes denote the stream of the forecast assimilation.

sider one ideal ensemble forecast, where the initial states of the ensemble members are normally distributed around the truth. The ensemble mean is the best estimation of truth because the mean coincides with the truth. The distribution of the ensemble members would be stretched by a linear error growth at the beginning of the forecast. Since the distribution is hyper cubic or elliptic, the ensemble mean is still the best forecast by the same reason. After that, the nonlinear effect becomes dominant in the nonlinear system. When the distribution is folded by the nonlinear effect, the ensemble mean is detached from the center of the distribution of the ensemble members. To avoid such detachment, we introduce the forecast assimilation in which the true value contained in the ensemble members is accumulated into a single forecast using a data assimilation technique. Generally, the initial errors are superimposed on the control run, since the truth is unknown for the ensemble members. Nevertheless, comparing the forecast skill under the known truth may be the first step to show the usefulness of the assimilated forecast.

Figure 1 illustrates a schematic flowchart of the forecast assimilation. The bold arrows and boxes denote the stream of the forecast assimilation. First, a control forecast and some ensemble forecasts are calculated from the initial state at t_0. Since we cannot know the true initial state, the initial state of the control and ensemble forecast contain unavoidable error around the true state. As the time integration proceeds, the forecast error would develop from the initial error in spite of the perfect model setting. At t_i, some time after the beginning, we start the forecast assimilation. Here, the ensemble members at t_i are regarded as the observations corresponding to the ordinary data assimilation. We will refer to it as predicted observations. Then the predicted observations are assimilated into the background of the control forecast using the Kalman filter, and analyzed value is obtained by the forecast assimilation. At the next time step t_{i+1}, the next predicted observation is assimilated into the analyzed background. The forecast assimilation is calculated every time step after the time t_{i+1}. This new type of forecast is defined as assimilated forecast. In this experiment, we compare the forecast skill among control forecast, ensemble mean, and the assimilated forecast at any time t_n.

2.2 Lorenz model

In order to understand the features of the forecast assimilation, we consider the Lorenz model (Lorenz 1963), which has been studied by the comprehension of chaotic behavior and a nonlinear system (Sparrow 1982; Mukougawa et al. 1991). The Lorenz model consists of three differential equations,

$$\frac{dx}{dt} = -\sigma(x - y),$$
$$\frac{dy}{dt} = xz + \gamma x - y, \tag{1}$$
$$\frac{dz}{dt} = xy - \beta z,$$

where σ, γ, and β are the model parameters, which are chosen $\sigma = 10$, $\gamma = 28$, and $\beta = 8/3$. The Lorenz model has been used in many previous studies of data assimilation and ensemble prediction (Palmer 1993; Miller et al. 1994; Evensen 1997; Anderson and Anderson 1999).

2.3 Kalman filter

In order to examine the features of the forecast assimilation, we use a Kalman filter, which many laboratories conduct experiments of the data assimilation for weather forecasting (Bouttier and Courtier 1999; Anderson 2001; Hamill et al. 2001). Many operational centers are using three or four dimensional variational data assimilation systems, called 3D-var or 4D-var, for weather forecasting (Courtier et al. 1998; Rabier et al. 2000). It is shown that the result of the forecast assimilation using 3D-var results in the ensemble mean, as described in the Appendix. On the other hand, the Kalman filter includes an explicit description of the evolution of forecast error covariance in a data assimilation cycle, so the Kalman filter is superior to the variational analysis for data assimilation. Therefore, we expect high performance of forecast assimilation using the Kalman filter. Especially, the Kalman filter in the nonlinear system is called the extended Kalman filter, which we utilize for the forecast assimilation experiments in this study.

A detailed description of the extended Kalman filter is provided by Daley (1991) or Bouttier and Courtier (1999), so only a brief description is presented here. For the extended Kalman filter, a nonlinear forecast model is required. Then a vector of a forecast $\mathbf{x}_f$ at time $i+1$ is predicted, using the nonlinear forecast model M and a vector of an analysis $\mathbf{x}_a$ at time i:

$$\mathbf{x}_f(i+1) = M(\mathbf{x}_a(i)). \tag{2}$$

At the same time, we have some vectors which correspond to the observations $\mathbf{x}_o$ that have the same dimension as $\mathbf{x}_f$ and $\mathbf{x}_a$. In the forecast assimilation, $\mathbf{x}_o$ are calculated as the ensemble forecast by the same forecast model M rather than the true observation. Next, new $\mathbf{x}_a$ is then obtained using $\mathbf{x}_f$ and $\mathbf{x}_o$ by means of the following equation:

$$\mathbf{x}_a(i) = \mathbf{x}_f(i) + \mathbf{K}(i)(\mathbf{x}_o(i) - H(i)\mathbf{x}_f(i)), \tag{3}$$

where i is observation time, H is a observation operator, and $\mathbf{K}$ is the Kalman gain matrix given by

$$\mathbf{K}(i) = \mathbf{P}_f(i)\mathbf{H}^T(i)(\mathbf{H}(i)\mathbf{P}_f(i)\mathbf{H}^T(i) + \mathbf{R}(i))^{-1}. \tag{4}$$

Here, $\mathbf{P}_f$ is a forecast error covariance matrix, $\mathbf{R}$ is an observation error covariance matrix, and $\mathbf{H}$ is a tangent linear matrix of the observational operator H in the vicinity of $\mathbf{x}_f$. In this experiment, the observations are the same variables in the model, so H and $\mathbf{H}$ are equal to an identity matrix $\mathbf{I}$. Therefore, Eq. (3) and Eq. (4) are rewritten as

$$\mathbf{x}_a(i) = \mathbf{x}_f(i) + \mathbf{K}(i)(\mathbf{x}_o(i) - \mathbf{x}_f(i)), \tag{5}$$
$$\mathbf{K}(i) = \mathbf{P}(i)_f(\mathbf{P}(i)_f + \mathbf{R}(i))^{-1}. \tag{6}$$

The observation error covariance matrix $\mathbf{R}$ may be defined by a difference between $\mathbf{x}_o$ and true state $\mathbf{x}_t$ by

$$\mathbf{R} = \overline{(\mathbf{x}_o - \mathbf{x}_t)(\mathbf{x}_o - \mathbf{x}_t)^T}, \tag{7}$$

where the overbar denotes an expectation value. The $\mathbf{P}_f$ is predicted for the next time step using the model, and given by the next two equations,

$$\mathbf{P}_a(i) = (\mathbf{I} - \mathbf{K}(i))\mathbf{P}_f(i), \tag{8}$$
$$\mathbf{P}_f(i+1) = \mathbf{M}(i)\mathbf{P}_a(i)\mathbf{M}^T(i) + \mathbf{Q}(i), \tag{9}$$

where $\mathbf{P}_a$ is an analysis error covariance matrix, and Eq. (9) indicates the forecast of $\mathbf{P}_f$ using the tangent linear model matrix $\mathbf{M}$ of the nonlinear forecast model M with a model error covariance matrix $\mathbf{Q}$.

Eqs. (2) and (9) are the prediction portion of the extended Kalman filter, and Eqs. (5), (6) and (8) are the analysis portion. If $\mathbf{P}_f(t_i), \mathbf{R}(t_i)$, $\mathbf{Q}(t_i)$, and $\mathbf{x}_o(t_i)$ are determined at the first step of the forecast assimilation in Fig. 1, $\mathbf{x}_a$ can be routinely calculated for every time step of the forecast model. Now, the $\mathbf{x}_o(t_i)$ is randomly chosen among the ensemble members. $\mathbf{P}_f(t_i)$ is assumed as $\mathbf{P}_f(t_i) = \mathbf{R}(t_i)$, because the distance between the control forecast and the truth equals the average distance between each ensemble member and the truth. Furthermore, we assume a perfect model for M, so $\mathbf{Q} = 0$. Unfortunately, it is impossible to obtain $\mathbf{R}(t_i)$ in Eq. (7), because the future of the true value is unknown to us. Therefore, we consider that $\mathbf{R}(t_i)$ is obtained from an ensemble Kalman filter technique (Evensen 1994; Burgers et al. 1998), which uses an ensemble forecast to estimate $\mathbf{R}(t_i)$. In the ensemble Kalman filter, the ensemble covariance matrix $\mathbf{R}_e(i)$ is used with the ensemble mean $\overline{\mathbf{x}_e}$ for the substitute of $\mathbf{x}_t$

$$\mathbf{R}(i) \simeq \mathbf{R}_e(i) = \overline{(\mathbf{x}_o(i) - \overline{\mathbf{x}_e(i)})(\mathbf{x}_o(i) - \overline{\mathbf{x}_e(i)})^T}. \quad (10)$$

In the data assimilation, since we cannot know the truth, $\mathbf{R}$ is calculated by the previous $\mathbf{x}_o$, and the analysis value instead of $\mathbf{x}_t$. Similarly, since $\overline{\mathbf{x}_e}$ is a better estimation of the truth than the control forecast or ensemble members in the future state, the $\mathbf{R}_e(i)$ is considered as the $\mathbf{R}(i)$. Therefore, the assimilated forecast is continuously calculated by the forecast assimilation system using the ensemble members.

3. Results

First, a true run is integrated using the Lorenz model with an initial state given by $(x_0, y_0, z_0) = (1.508870, -1.531271, 25.46091)$ for time $t = 0$ to $t = 50000.0$ with 0.001 time step. The true run is divided into 5000 sectors for every $\Delta t = 10.0$ (denoted as 0.0 T to 10.0 T). The examination of the forecast assimilation is carried out for every sector. A control forecast and 200 members of the ensemble forecast are integrated from the start point of each sector of the true run adding Gaussian noise with zero mean and variance equals to 0.0025. First, the observation error covariance matrix $\mathbf{R}$ at Eq. (10) is calculated from the 100 members of the ensemble forecast. The remaining 100 members are utilized for the forecast assimilation.

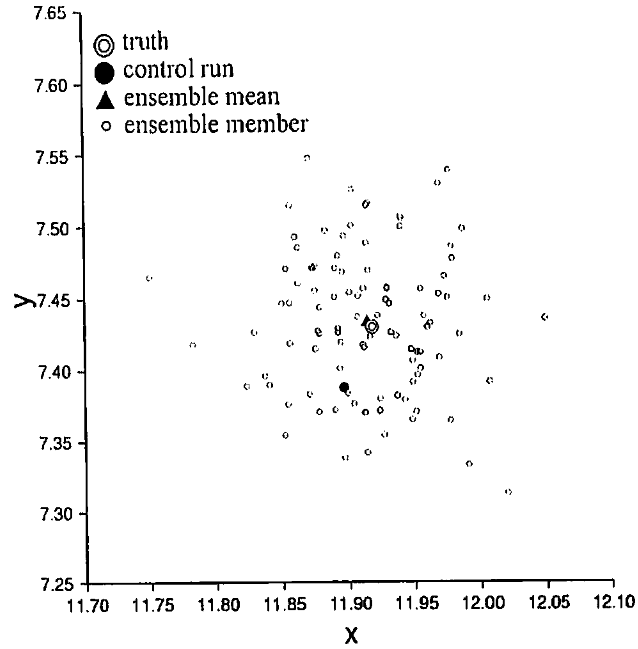

Fig. 2. Initial distribution of the true run, control forecast, ensemble members, and the ensemble mean on $x - y$ plane at 0.0 T. The ensemble members are normally distributed around the truth.

Figure 2 shows the initial distribution of the true run, control forecast, 100 members of the ensemble forecast, and the ensemble mean on $x - y$ plane at 0.0 T for an example of a sector. The average root mean square error (RMSE) of the control, and each ensemble member against the true run is 0.05. Nevertheless, the RMSE of the ensemble mean is nearly equal to zero, because the distribution of ensemble members is Gaussian around the true run. As time proceeds, the ensemble members diverge by the linear and nonlinear effects of the Lorenz model, so the RMSE of the ensemble members exponentially increases, and the unimodal distribution evolves into a bimodal distribution.

As the first example, we describe one result of the forecast assimilation on a sector with a good forecast skill. The forecast assimilation is started at 3.5 T with 0.001 time step, and a predicted observation is randomly chosen from the 100 ensemble members. Figure 3 illustrates forecast distributions of the true run, control forecast, ensemble members, and the ensemble mean on $x - y$ plane at 3.5 T. The distribution of the ensemble members describes an arc, and the true run and the control forecast lie on the

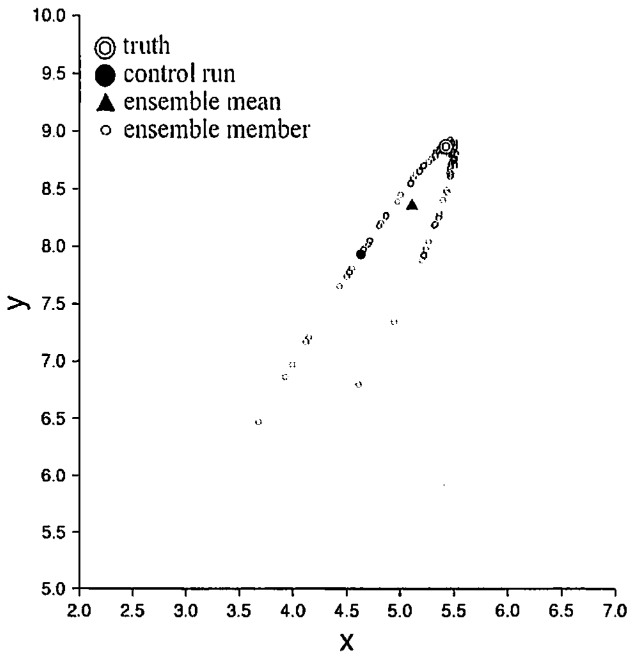

Fig. 3. Same as Fig. 2, but starting point of the forecast assimilation at 3.5 T. The distribution of the forecasts describes an arc.

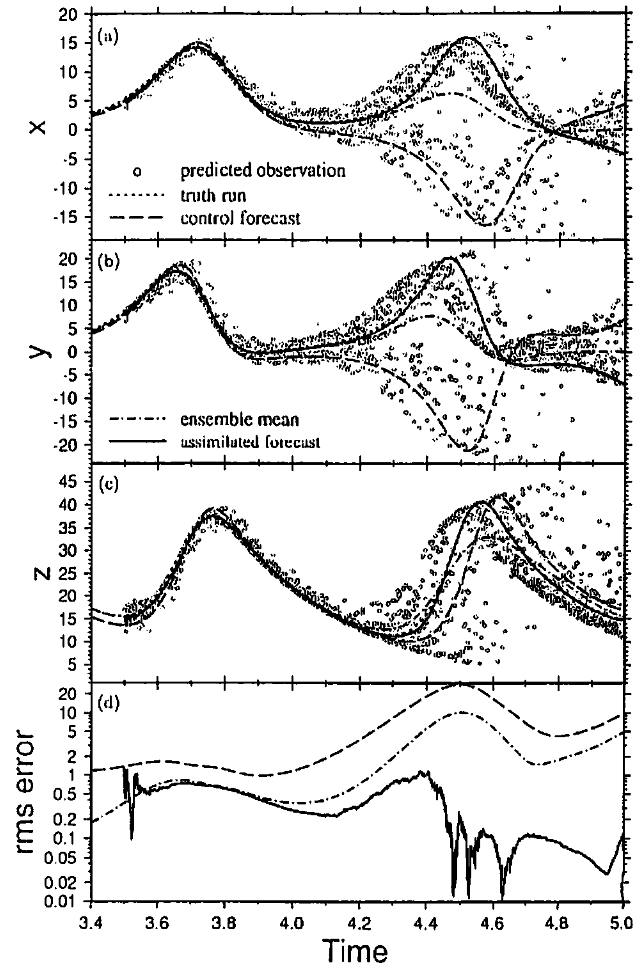

Fig. 4. Sample of one result of the truth, control forecast, ensemble mean, predicted observation, and assimilated forecast. The forecast assimilation is started at 3.5 T; (a) time variation of value of x variable, (b) y variable, (c) z variable, and (d) RMSE for control forecast, ensemble mean, and assimilated forecast.

arc. The ensemble mean, however, is located apart from the arc.

The first step of the forecast assimilation is that the randomly chosen ensemble member is assimilated into the control forecast in Fig. 1. Figure 4 illustrates trajectories and RMSE of the truth, control forecast, ensemble mean, and assimilated forecast on the sector examined. At the beginning of the forecast assimilation, the trajectories of all forecasts are near the truth. The RMSE of the assimilated forecast fluctuates because of the effect of the forecast assimilation. But it decreases to the error level lower than the ensemble mean and becomes stable at 3.6 T. Figure 5 illustrates the early evolution of the forecast distributions of the truth, control forecast, ensemble members, the ensemble mean, and assimilated forecast on $x-y$ plane. The assimilated forecast is located in the neighborhood of the ensemble mean, but slightly closer to the truth.

After time 4.0 T in Fig. 4, the distribution of the predicted observations spreads widely around the truth, and the control forecast is apart continuously from the truth. Then, the fluctuation of the assimilated forecast is increasing for spread observations from Eq. (5) and reinforcement of the nonlinearity in the model states from Eqs. (2) and (9). Figure 6 plots the time variation of a norm of the Kalman gain matrix. The range of the forecast time is the same as in Fig. 4. When the norm of the Kalman gain is large, mixing ratio of the predicted observation into the background is increasing from Eq. (6). Although in Fig. 4 the assimilated forecast accumulates the observations with large error, the trajectory of the assimilated forecast is close to the truth, and the RMSE becomes one or two order smaller than the other forecasts. Figure 7 illustrates continuation of Fig. 5. A unimodal distribution of the ensemble members at 4.0 T changes to bimodal at 4.5 T. Then, the ensemble mean is seen at

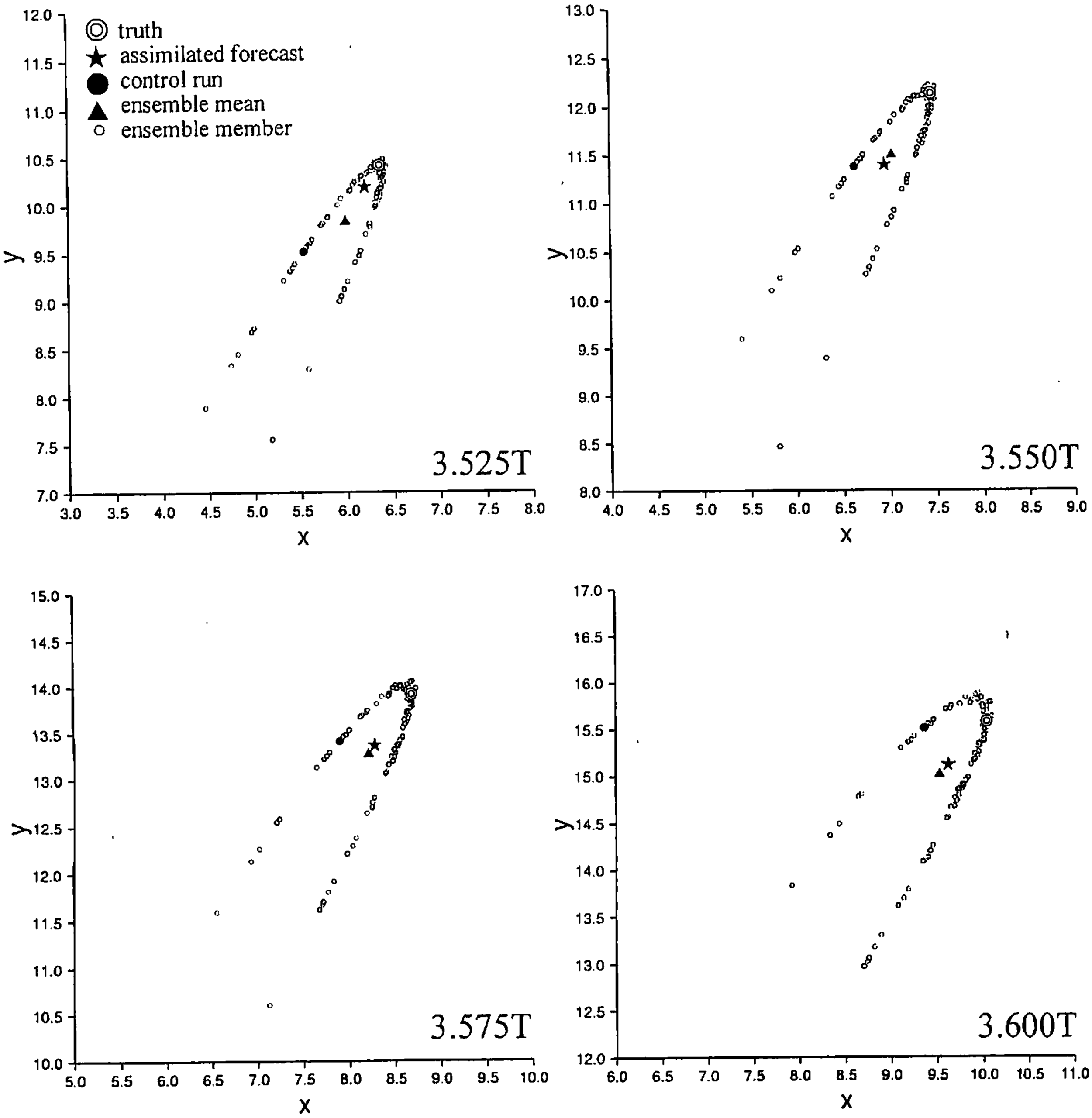

Fig. 5. Evolution of the forecast distributions of the true run, control forecast, ensemble members, the ensemble mean, and assimilated forecast on $x - y$ plane.

the center of the two distributions. Therefore, the RMSE of the ensemble mean is larger than the assimilated forecast at 4.5 T even if the ensemble mean is located in the neighborhood of the true run and the assimilated forecast at 4.0 T. On the other hand, the assimilated forecast that has obtained some true value from some predicted observations lays on the true run at 4.5 T.

The forecast assimilation experiment is repeated for 5000 sectors, to increase the statistical confidence. Figure 8 illustrates averaged RMSE for 5000 samples of the control forecast, ensemble mean, and assimilated forecast. In this case, it is indicated that the averaged RMSE of each ensemble member is comparable to the error level of the control forecast. The RMSE of all forecasts, except for the assimilated forecast, are exponentially growing at the early stage, and the error gradually saturates by the nonlinear effect. In this situation, all forecast assimilations in every sector are started at 3.5 T. The RMSE of the assimilated forecast, which has the same error as the con-

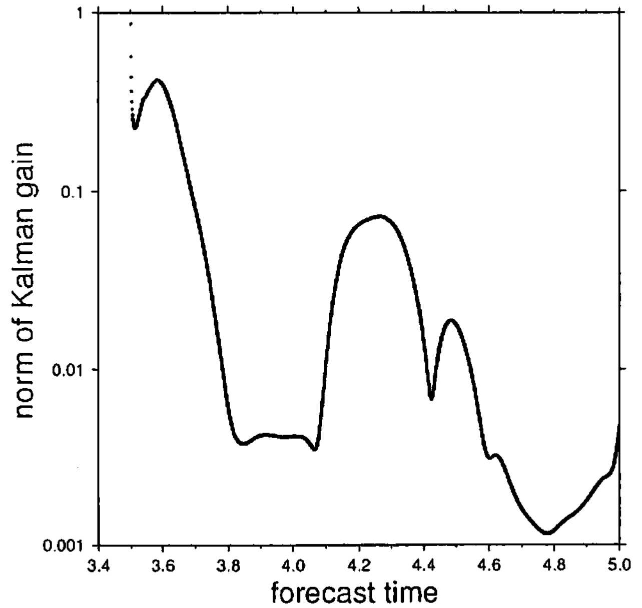

Fig. 6. Norm of Kalman gain matrix. The range of the forecast time is the same as Fig. 4.

trol forecast at the starting time of assimilation, decreases below the ensemble mean in a short time. After 3.8 T, the RMSE of the assimilated forecast is exponentially increasing as in the control forecast or ensemble mean. At 6.0 T, the RMSE of the assimilated forecast exceeds the ensemble mean. When the RMSE of the control forecast, or ensemble members approach to the saturated level, the predicted observation used by the forecast assimilation has little positional information of the true state. Therefore, the error of the assimilated forecast close to the saturated level grows more rapidly than the normal forecast.

The effect of the forecast assimilation depends on the distribution of the ensemble members. Immediately after the start of the ensemble forecast, the distribution is similar to Gaussian for the linear error growth. If the forecast assimilation starts earlier than 3.5 T, it is expected that the forecast assimilation searches more accurate value of the truth because the ensemble members are close to the normal distribution that is favorable to the Kalman filter. Likewise, the state of the ensemble mean is significantly close to the truth for hyper elliptic distribution. Therefore, it is necessary to compare the forecast skill of the assimilated forecast with the ensemble mean in various starting times of the forecast assimilation.

Figure 9 illustrates the ratio of RMSE of the assimilated forecast against the ensemble mean. The abscissa indicates the forecast time and the ordinate indicates the starting time of the forecast assimilation. The area where the assimilated forecast is more skillful than the ensemble mean is shaded (the ratio is smaller

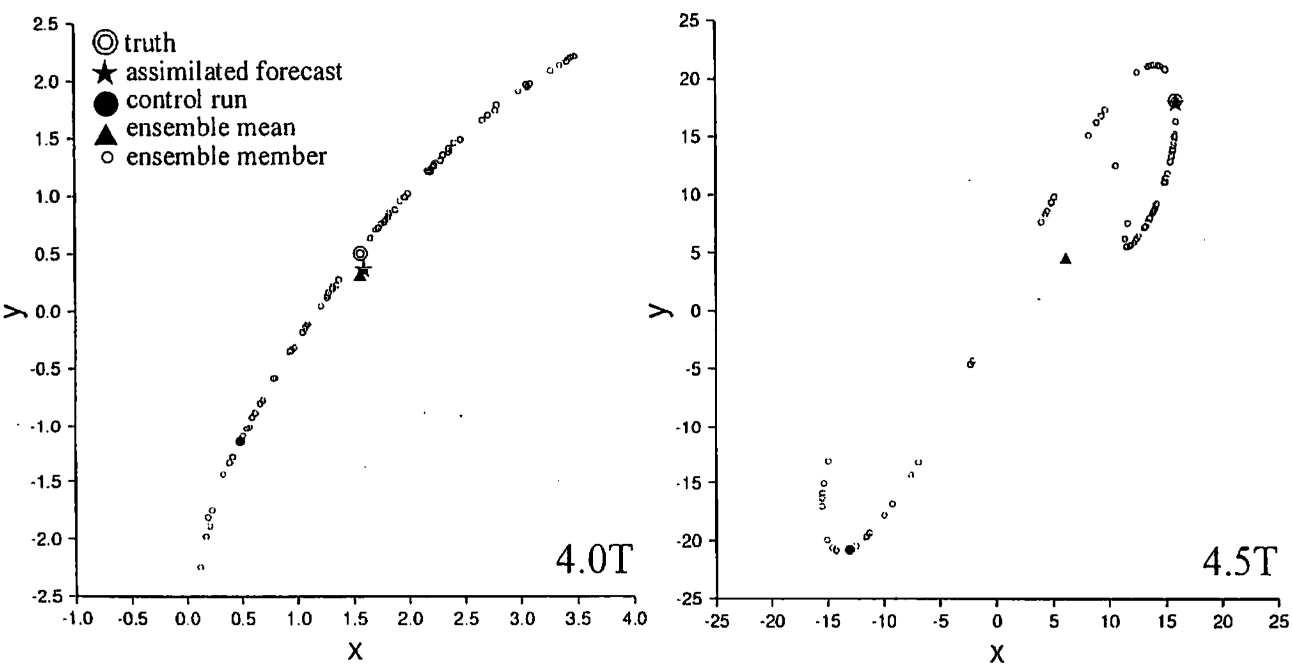

Fig. 7. Continuation of Fig. 5 for 4.5 T. But the scale unit of axes in (b) is changed to ten times since the forecast distributions are spread.

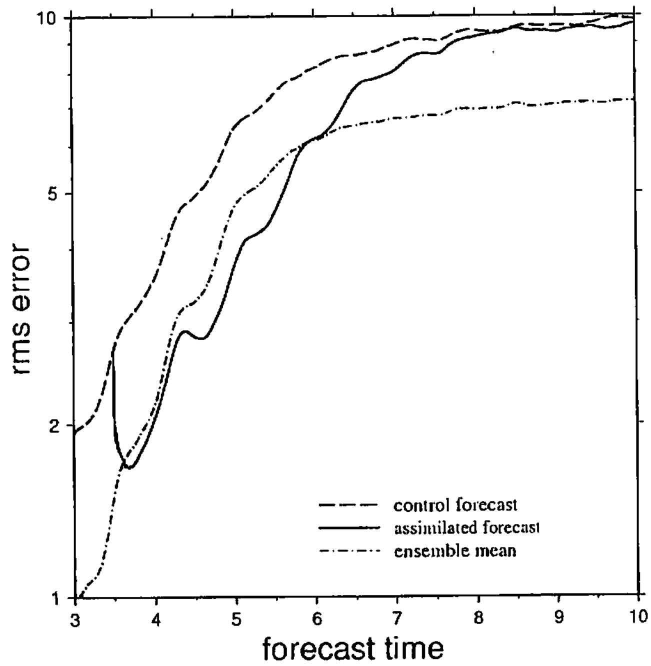

Fig. 8. Averaged RMSE for 5000 samples of the control forecast, ensemble mean, and assimilated forecast. The forecast assimilation is started at time 3.5 T.

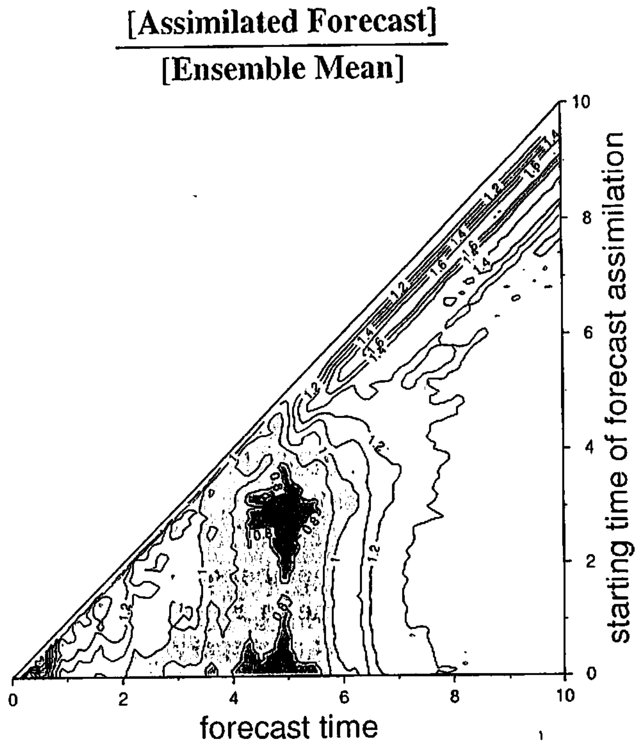

Fig. 9. Ratio of RMSE of the assimilated forecast to the ensemble mean. The abscissa indicates the forecast time and the ordinate indicates the starting time of the forecast assimilation. The area where the assimilated forecast is more skillful than the ensemble mean is shaded (the ratio is smaller than 1).

than 1). In this result, the error level of the ensemble mean as time proceeds is comparable to RMSE of the ensemble mean in Fig. 8. Similarly, the average error of each ensemble member is comparable to RMSE of the control forecast in Fig. 8. From 3.0 T to 6.0 T forecast time, the forecast skill of the assimilated forecast is superior to the ensemble mean irrespective of the starting time of the forecast assimilation. The distribution of the ensemble members is slightly folded by the nonlinear effect, e.g. Fig. 3, so the ensemble mean is detached from the center of the distribution of the ensemble members. The assimilated forecast is, however, inferior before 3.0 T, since the ensemble mean is located near the truth by the assumption of the error distribution centered around the truth. In addition, it is inferior after 6.0 T, since the ensemble members have little positional information of truth, because of saturated forecast error of the ensemble members. Nevertheless, the assimilated forecast has a good performance at the intermediate, where the nonlinear growth dominates but is not saturated.

4. Summary and discussion

In this study a new type of ensemble forecast assimilation technique is developed in order to improve the forecast skill in the nonlinear dynamical system. The forecast assimilation is an analysis technique in which true value contained in each ensemble forecast is accumulated into a single assimilated forecast such as a data assimilation. For the experiments, we used a Lorenz model, and a Kalman filter is applied for the forecast assimilation.

The experiments are started by calculating 101 members of the ensemble forecast in which the initial error with Gaussian distribution is superimposed around the true run, and one of the members is arbitrarily selected as a control forecast. The experiments of the forecast assimilation are repeated 5000 times for different sectors of the solution trajectory to obtain the statistical significance of the results. The distribution of the ensemble members is stretched by a linear error growth at the beginning of the forecast. After that, the nonlinear effect becomes dominant to distort the distribution. The forecast assimilation is then started, when the errors of the ensemble forecasts have grown to a certain threshold.

It is demonstrated that the forecast skill of the assimilated forecast is always superior to the control forecast. In the range of the small RMSE of the ensemble forecasts (e.g. 0.0 T to 3.0 T forecast time in Fig. 9), the skill of the assimilated forecast is inferior to the ordinary ensemble mean. Since the distribution of the ensemble forecasts is similar to the hyper ellipsoid until the nonlinear effect becomes dominant, the center of the distribution is always close to the truth. After the distribution is folded by the nonlinear effect, e.g. Fig. 3, the ensemble mean is detached from the distribution of the ensemble members. From 3.0 T to 6.0 T of the forecast time in Fig. 9, the skill of the assimilated forecast is superior to the ensemble mean. After 6.0 T forecast time in Fig. 9, the skill of the assimilated forecast is poorer than the ensemble mean, since the ensemble members have little positional information of the truth for saturated forecast errors of the ensemble members. Nevertheless, the assimilated forecast has a good performance at the intermediate, where the nonlinear growth dominates but is not saturated.

The reasons for the superior performance of the assimilated forecast may be explained by the following characteristics of the Kalman filer. (1): Immediately after the starting point of the forecast assimilation, the assimilated forecast searches the true value contained in the ensemble members because $P(0) = R(0)$ has been assumed at the beginning. In this range the Kalman filter can quickly reduce the error of the assimilated forecast. (2): Then, in the direction of the error growth of the linearized Lorenz model, the Kalman filter can quickly reduce the error because $P(i)$ contains information about the unstable direction by its history. Conversly, near the saturation of the error as seen in Fig. 9 at time larger than 6.0 T, the Kalman filter rather increases the error by assimilating ensemble members without information about the truth.

Based on the above remarks, let us assume that the distribution of the ensemble members has been separated in two groups by the dynamical instability. Then the ensemble mean chooses just the center of the two separated groups regardless of the stability of the separated trajectories. In contrast, the trajectory of the assimilated forecast randomly chooses one of the two groups because the predicted observation is randomly selected. After the branch point, we assume that there are two results of the assimilated forecast: one becomes stable trajectory, and the other becomes unstable trajectory. In the former (latter) case, the assimilated forecast absorbs relatively less (much) positional information of the predicted observations that are included in the two groups. Repeating the forecast assimilations, the trajectory of the assimilated forecast in the former (latter) case becomes smooth (fluctuative), and it is difficult (easy) to shift to the unstable (stable) trajectory. Therefore, the assimilated forecast tends to move from the unstable to the stable trajectory. Since the stable trajectory is one of the most suitable solutions in the ensemble forecast, the forecast skill becomes superior to the ensemble mean in the nonlinear regime.

One point to notice in our examination is that the Gaussian distribution of the perturbations has been assumed around the true initial state for the ensemble members. Therefore, the mean of the ensemble members knows the true value at the beginning. After the nonlinear effect of the forecast error is dominated, the forecast assimilation searches for the truth better than the ensemble mean. If the analysis errors and model errors are cancelled as expected in the multi-analysis multi-model, the forecast assimilation would be one of the viable approaches to the medium or extended range forecast.

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Appendix

Forecast assimilation using 3D-var

A detail description of the 3D-var is provided by Daley (1997) or Kalnay (2002), so only a brief description is presented in the following.

The principle of the 3D-var is to avoid the computation of Eq. (6) in the Kalman gain $\mathbf{K}$, by looking for the analysis as an approximate solution to the equivalent minimization problem defined by the cost function J as

$$J(\mathbf{x}) = \frac{1}{2}((\mathbf{x}-\mathbf{x}_f)^T\mathbf{P}_f^{-1}(\mathbf{x}-\mathbf{x}_f) + (\mathbf{x}-\mathbf{x}_o)^T\mathbf{R}^{-1}(\mathbf{x}-\mathbf{x}_o)), \quad (11)$$

where $\mathbf{P}_f$ and $\mathbf{R}$ are forecast and observation error covariance matrices, and $\mathbf{x}_f$ and $\mathbf{x}_o$ are vectors of control forecast and observation. For the forecast assimilation, the ensemble member $\mathbf{x}_i$ is regarded as the predicted observation $\mathbf{x}_o$. Since an average error of individual ensemble members from the truth theoretically equals an error of the control forecast $\mathbf{x}_f$, it is considered that $\mathbf{R}$ equals $\mathbf{P}_f$. If we use all ensemble forecasts (n members) for the forecast assimilation at the same time, the cost function J is given by

$$J(\mathbf{x}) = \frac{1}{2}\left((\mathbf{x}-\mathbf{x}_f)^T\mathbf{P}_f^{-1}(\mathbf{x}-\mathbf{x}_f) + \sum_{i=1}^{n}(\mathbf{x}-\mathbf{x}_i)^T\mathbf{P}_f^{-1}(\mathbf{x}-\mathbf{x}_i)\right). \quad (12)$$

The gradient of J is obtained by differentiating Eq. (12) with respect to $\mathbf{x}$,

$$\nabla J(\mathbf{x}) = \mathbf{P}_f^{-1}(\mathbf{x}-\mathbf{x}_f) + \sum_{i=1}^{n}\mathbf{P}_f^{-1}(\mathbf{x}-\mathbf{x}_i). \quad (13)$$

At the minimum of J, the gradient cost function of Eq. (13) leads to 0. Since $\mathbf{x}_f$ is considered as a zeroth ensemble member of $\mathbf{x}_i$, so Eq. (13) is rewritten by

$$\sum_{i=0}^{n}\mathbf{P}_f^{-1}(\mathbf{x}-\mathbf{x}_i) = 0. \quad (14)$$

Therefore, we obtain the best analysis $\mathbf{x}_a$ as

$$\mathbf{x}_a = \frac{\sum_{i=0}^{n}\mathbf{x}_i}{n+1}. \quad (15)$$

This equation indicates the ensemble mean. Therefore, the result of the forecast assimilation, using the 3D-var, results in the ensemble mean of the ensemble members.

References

Anderson, J.L., 2001: An ensemble adjustment Kalman filter for data assimilation. *Mon. Wea. Rev.*, **129**, 2884–2903.

———, S.L. Anderson, 1999: A Monte Carlo implementation of the nonlinear filtering problem to produce ensemble assimilation and forecasts. *Mon. Wea. Rev.*, **127**, 2741–2758.

Bouttier, F., P. Courtier, 1999: Data assimilation concepts and methods. *ECMWF Meteorological Training Course Letter Series*.

Buizza, R., J. Barkmeijer, T.N. Palmer, and D.S. Richardson, 2000: Current status and future developments of the ECMWF Ensemble Prediction System. *Meteorol. Appl.*, **7**, 163–175.

Burgers, G., P.J. van Leeuwen, and G. Evensen, 1998: Analysis scheme in the ensemble assimilation and forecasts. *Mon. Wea. Rev.*, **126**, 1719–1724.

Courtier, P., E. Andersson, W. Heckley, J. Pailleux, D. Vasiljević, M. Hamrud, A. Hollingsworth, F. Rabier, and M. Fisher, 1998: The ECMWF implementation of three-dimensional variational assimilation (3D-Var). I: Formulation. *Quart. J. Roy. Meteor. Soc.*, **124**, 1783–1807.

Daley, R., 1991: *Atmospheric data analysis*. Cambridge University Press, 457 pp.

———, 1997: Atmospheric data assimilation. *J. Meteor. Soc. Japan*, **75**, 319–329.

Errico, R.M., R. Langland, and D.P. Baumhefner, 2002: The workshop in atmospheric predictability. *Bull. Amer. Meteor. Soc.*, **83**, 1341–1343.

Evensen, G., 1994: Sequential data assimilation with a nonlinear quasigeostrophic model using Monte Carlo methods to forecast error statistics. *J. Geophys. Res.*, **99** (C5), 10143–10167.

———, 1997: Advanced data assimilation for strongly nonlinear dynamics. *Mon. Wea. Rev.*, **125**, 1342–1354.

Fritsch, J.M., J. Hilliker, J. Ross, and R.L. Vislocky, 2000: Model consensus. *Wea. Forecasting*, **15**, 571–582.

Hamill, T.M., J.S. Whitaker, and C. Snyder, 2001: Distance-dependent filtering of background error covariance estimates in an ensemble Kalman filter. *Mon. Wea. Rev.*, **129**, 2776–2790.

Kalman, R. and R. Bucy, 1961: New results in linear prediction and filtering theory Trans. AMSE, *J. Basic Eng.*, **83D**, 95–108.

Kalnay, E., 2002: *Atmospheric modeling data assimilation and predictability*. Cambridge University Press, 341 pp.

Krishnamurti, T.N., C.M. Kishtawal, T.E. LaRow, D.R. Bachiochi, Z. Zhanf, C.E. Willifor, S. Gadgil, and S. Surendran, 1999: Improved weather and seasonal climate forecasts from multimodel superensemble. *Science*, **285**, 1548–1550.

Lorenz, E.N., 1963: Deterministic nonperiodic flow. *J. Atmos. Sci.*, **20**, 130–141.

——, 1982: Atmospheric predictability experiments with a large numerical model. *Tellus*, **34**, 505–513.

Miller, R.N., M. Ghil, F. Gauthiez, 1994: Advanced data assimilation in strongly nonlinear dynamical systems. *J. Atmos. Sci.*, **51**, 1037–1056.

Molteni, F., R. Buizza, T.N. Palmar, and T. Petroliagis, 1996: The ECMWF ensemble prediction system: methodology and validation. *Quart. J. Roy. Meteor. Soc.*, **122**, 73–119.

Mukougawa, H., M. Kimoto, and S. Yoden, 1991: A relationship between local error growth and quasi-stationary states: case study in the Lorenz system. *J. Atmos. Sci.*, **48**, 1231–1237.

Nohara, D. and H.L. Tanaka, 2001: Logarithmic relation between the initial error and predictability for the barotropic component of the atmosphere. *J. Meteor. Soc. Japan*, **79**, 161–171.

Palmer, T.N., 1993: Extended range atmospheric prediction and the Lorenz model. *Bull. Amer. Meteor. Soc.*, **74**, 49–66.

——, Č. Branković, and D.S. Richardson, 2000: A probability and decision-model analysis of PROVOST seasonal multi-model ensemble integration. *Quart. J. Roy. Meteor. Soc.*, **126**, 2012–2033.

Rabier, F., H. Järvinen, E. Klinker, J.F. Mahfouf, and A. Simmons, 2000: The ECMWF operational implementation of four-dimensional variational assimilation. I: Experimental results with simplified physics, 2000: *Quart. J. Roy. Meteor. Soc.*, **126**, 1143–1170.

Sparrow, C., 1982: *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors*. Springer-Verlag, 269 pp.

Toth, Z., 1991: Estimate of atmospheric predictability by circulation analogs. *Mon. Wea. Rev.*, **119**, 65–72.

——— and E. Kalnay, 1993: Ensemble forecasting at NMC: The generation of perturbations. *Bull. Amer. Meteor. Soc.*, **74**, 2317–2330.

——— and ———, 1997: Ensemble forecasting at NCEP and the breeding method. *Mon. Wea. Rev.*, **125**, 3297–3319.